PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

**Intermediate Mathematics**

The UNIVERSITY of PLYMOUTH

# Gradients and Directional Derivatives

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The aim of this package is to provide a short self assessment programme for students who want to obtain an ability in vector calculus to calculate gradients and directional derivatives.

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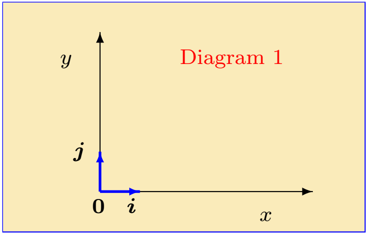
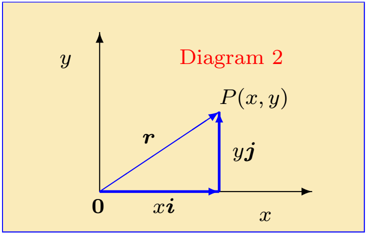
**The full range of these packages and some instructions, should they be required, can be obtained from our web page** Mathematics Support Materials.

# 1. Introduction (Vectors)

The base vectors in two dimensional Cartesian coordinates are the unit vector $\boldsymbol{i}$ in the positive direction of the $x$ axis and the unit vector $\boldsymbol{j}$ in the $y$ direction. See Diagram 1. (In three dimensions we also require $\boldsymbol{k}$, the unit vector in the $z$ direction.)

The position vector of a point $P(x, y)$ in two dimensions is $x\boldsymbol{i} + y\boldsymbol{j}$ . We will often denote this important vector by $\boldsymbol{r}$ . See Diagram 2. (In three dimensions the position vector is $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ .)



The vector differential operator $\boldsymbol{\nabla}$, called "del" or "nabla", is **defined** in three dimensions to be:

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\,.$$

Note that these are *partial derivatives*!
This vector operator may be applied to (differentiable) scalar functions (scalar fields) and the result is a special case of a vector field, called a gradient vector field.

Here are two warming up exercises on partial differentiation.

Quiz Select the following partial derivative, $\dfrac{\partial}{\partial z}\left(xyz^x\right)$.

(a) $x^2yz^{x-1}$, (b) $0$, (c) $xy\log_x(z)$, (d) $yz^{x-1}$.

Quiz Choose the partial derivative $\dfrac{\partial}{\partial x}\left(x\cos(y) + y\right)$.

(a) $\cos(y)$, (b) $\cos(y) - x\sin(y) + 1$,
(c) $\cos(y) + x\sin(y) + 1$, (d) $-\sin(y)$.

# 2. Gradient (Grad)

The **gradient** of a function, $f(x, y)$, in two dimensions is defined as:

$$\operatorname{grad} f(x,y) = \boldsymbol{\nabla} f(x,y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}\,.$$

The gradient of a function is a vector field. It is obtained by applying the vector operator $\boldsymbol{\nabla}$ to the scalar function $f(x, y)$. Such a vector field is called a gradient (or conservative) vector field.

**Example 1** The gradient of the function $f(x, y) = x + y^2$ is given by:

$$\begin{aligned}
\boldsymbol{\nabla} f(x,y) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} \\
&= \frac{\partial}{\partial x}(x+y^2)\boldsymbol{i} + \frac{\partial}{\partial y}(x+y^2)\boldsymbol{j} \\
&= (1+0)\boldsymbol{i} + (0+2y)\boldsymbol{j} \\
&= \boldsymbol{i} + 2y\boldsymbol{j}\,.
\end{aligned}$$

Quiz Choose the gradient of $f(x, y) = x^2 y^3$.

(a) $2x\boldsymbol{i} + 3y^2\boldsymbol{j}$ , (b) $x^2\boldsymbol{i} + y^3\boldsymbol{j}$ ,

(c) $2xy^3\boldsymbol{i} + 3x^2y^2\boldsymbol{j}$ , (d) $y^3\boldsymbol{i} + x^2\boldsymbol{j}$ .

The definition of the **gradient** may be extended to functions defined in three dimensions, $f(x, y, z)$:

$$\boldsymbol{\nabla} f(x, y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\,.$$

Exercise 1. Calculate the gradient of the following functions (click on the **green** letters for the solutions).

(a) $f(x, y) = x + 3y^2$ , (b) $f(x, y) = \sqrt{x^2 + y^2}$ ,

(c) $f(x, y, z) = 3x^2\sqrt{y} + \cos(3z)$ , (d) $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ ,

(e) $f(x, y) = \dfrac{4y}{(x^2 + 1)}$ , (f) $f(x, y, z) = \sin(x)e^y \ln(z)$ .

# 3. Directional Derivatives

To interpret the gradient of a scalar field

$$\boldsymbol{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\,,$$

note that its component in the $\boldsymbol{i}$ direction is the partial derivative of $f$ with respect to $x$. This is the rate of change of $f$ in the $x$ direction since $y$ and $z$ are kept constant. In general, the component of $\boldsymbol{\nabla} f$ in any direction is the rate of change of $f$ in that direction.

**Example 2** Consider the scalar field $f(x,y) = 3x + 3$ in two dimensions. It has no $y$ dependence and it is linear in $x$. Its gradient is given by

$$\begin{aligned} \boldsymbol{\nabla} f &= \frac{\partial}{\partial x}(3x+3)\boldsymbol{i} + \frac{\partial}{\partial y}(3x+3)\boldsymbol{j} \\ &= 3\boldsymbol{i} + 0\boldsymbol{j}\,. \end{aligned}$$

As would be expected the gradient has zero component in the $y$ direction and its component in the $x$ direction is constant (3).

Quiz Select a point from the answers below at which the scalar field $f(x, y, z) = x^2yz - xy^2z$ *decreases* in the $y$ direction.

(a) $(1, -1, 2)$, (b) $(1, 1, 1)$,

(c) $(-1, 1, 2)$, (d) $(1, 0, 1)$.

**Definition:** if $\hat{\boldsymbol{n}}$ is a unit vector, then $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} f$ is called the directional derivative of $f$ in the direction $\hat{\boldsymbol{n}}$. The directional derivative is the rate of change of $f$ in the direction $\hat{\boldsymbol{n}}$.

**Example 3** Let us find the directional derivative of $f(x, y, ) = x^2yz$ in the direction $4\boldsymbol{i} - 3\boldsymbol{k}$ at the point $(1, -1, 1)$.
The vector $4\boldsymbol{i} - 3\boldsymbol{k}$ has magnitude $\sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$. The unit vector in the direction $4\boldsymbol{i} - 3\boldsymbol{k}$ is thus $\hat{\boldsymbol{n}} = \frac{1}{5}(4\boldsymbol{i} - 3\boldsymbol{k})$.
The gradient of $f$ is

$$\begin{aligned} \boldsymbol{\nabla} f &= \frac{\partial}{\partial x}(x^2yz)\boldsymbol{i} + \frac{\partial}{\partial y}(x^2yz)\boldsymbol{j} + \frac{\partial}{\partial z}(x^2yz)\boldsymbol{k} \\ &= 2xyz\boldsymbol{i} + x^2z\boldsymbol{j} + x^2y\boldsymbol{k}\,, \end{aligned}$$

and so the required directional derivative is

$$\begin{aligned}\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla} f &= \frac{1}{5}(4\boldsymbol{i}-3\boldsymbol{k})\cdot(2xyz\boldsymbol{i}+x^2z\boldsymbol{j}+x^2y\boldsymbol{k})\\ &= \frac{1}{5}\left[4\times 2xyz+0-3\times x^2y\right].\end{aligned}$$

*At the point* $(1,-1,1)$ the desired directional derivative is thus

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla} f=\frac{1}{5}\left[8\times(-1)-3\times(-1)\right]=-1\,.$$

EXERCISE 2. Calculate the directional derivative of the following functions in the given directions and at the stated points (click on the **green** letters for the solutions).

(a) $f=3x^2-3y^2$ in the direction $\boldsymbol{j}$ at (1,2,3).

(b) $f=\sqrt{x^2+y^2}$ in the direction $2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$ at (0,−2,1).

(c) $f=\sin(x)+\cos(y)+\sin(z)$ in the direction $\pi\boldsymbol{i}+\pi\boldsymbol{j}$ at ($\pi$,0,$\pi$ ).

We now state, without proof, **two useful properties** of the directional derivative and gradient.

- The maximal directional derivative of the scalar field $f(x, y, z)$ is in the direction of the gradient vector $\boldsymbol{\nabla} f$.
- If a surface is given by $f(x, y, z) = c$ where $c$ is a constant, then the normals to the surface are the vectors $\pm\boldsymbol{\nabla} f$.

**Example 4** Consider the surface $xy^3 = z + 2$. To find its unit normal at $(1, 1, -1)$, we need to write it as $f = xy^3 - z = 2$ and calculate the gradient of $f$:

$$\boldsymbol{\nabla} f = y^3\boldsymbol{i} + 3xy^2\boldsymbol{j} - \boldsymbol{k}\,.$$

At the point $(1, 1, -1)$ this is $\boldsymbol{\nabla} f = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$. The *magnitude* of this maximal rate of change is

$$\sqrt{1^2 + 3^2 + (-1)^2} = \sqrt{11}\,.$$

Thus the unit normals to the surface are $\pm\dfrac{1}{\sqrt{11}}(\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$.

Quiz Which of the following vectors is normal to the surface $x^2yz = 1$ at (1,1,1)?

(a) $4\boldsymbol{i} + \boldsymbol{j} + 17\boldsymbol{k}\,,$ (b) $2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}\,,$

(c) $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}\,,$ (d) $-2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}\,.$

Quiz Which of the following vectors is a unit normal to the surface $\cos(x)yz = -1$ at ($\pi$,1,1)?

(a) $-\frac{1}{\sqrt{2}}\boldsymbol{j} + \frac{1}{\sqrt{2}}\boldsymbol{k}\,,$ (b) $\pi\boldsymbol{i} + \boldsymbol{j} + \frac{2}{\sqrt{\pi}}\boldsymbol{k}\,,$

(c) $\boldsymbol{i}\,,$ (d) $-\frac{1}{\sqrt{2}}\boldsymbol{j} - \frac{1}{\sqrt{2}}\boldsymbol{k}\,.$

Quiz Select a unit normal to the (spherically symmetric) surface $x^2 + y^2 + z^2 = 169$ at (5,0,12).

(a) $\boldsymbol{i} + \frac{1}{6}\boldsymbol{j} - \frac{1}{6}\boldsymbol{k}\,,$ (b) $\frac{1}{3}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j} + \frac{1}{3}\boldsymbol{k}\,,$

(c) $\frac{5}{13}\boldsymbol{i} + \frac{12}{13}\boldsymbol{k}\,,$ (d) $-\frac{5}{13}\boldsymbol{i} + \frac{12}{13}\boldsymbol{k}\,.$

# 4. Final Quiz

Begin Quiz Choose the solutions from the options given.

**1.** What is the gradient of $f(x, y, z) = xyz^{-1}$?

(a) $\boldsymbol{i} + \boldsymbol{j} - z^{-2}\boldsymbol{k}$,
(b) $\dfrac{y}{z}\boldsymbol{i} + \dfrac{x}{z}\boldsymbol{j} - \dfrac{xy}{z^2}\boldsymbol{k}$,
(c) $yz^{-1}\boldsymbol{i} + xz^{-1}\boldsymbol{j} + xyz^{-2}\boldsymbol{k}$,
(d) $-\dfrac{1}{z^2}$.

**2.** If $n$ is a constant, choose the gradient of $f(\boldsymbol{r}) = 1/r^n$, where $r = |\boldsymbol{r}|$ and $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

(a) $0$,
(b) $-\dfrac{n}{2}\dfrac{\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}}{r^{n+1}}$,
(c) $-\dfrac{n\boldsymbol{r}}{r^{n+2}}$,
(d) $-\dfrac{n}{2}\dfrac{\boldsymbol{r}}{r^{n+2}}$.

**3.** Select the unit normals to the surface of revolution, $z = 2x^2 + 2y^2$ at the point (1,1,4).

(a) $\pm\dfrac{1}{\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$,
(b) $\pm\dfrac{1}{\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$,
(c) $\pm\dfrac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{j})$,
(d) $\pm\dfrac{1}{\sqrt{2}}(2\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k})$.

End Quiz Score: Correct

# Solutions to Exercises

**Exercise 1(a)** The function $f(x,y) = x + 3y^2$ , has gradient

$$\begin{aligned}
\boldsymbol{\nabla} f(x,y) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} \\
&= \frac{\partial}{\partial x}(x+3y^2)\boldsymbol{i} + \frac{\partial}{\partial y}(x+3y^2)\boldsymbol{j} \\
&= (1+0)\boldsymbol{i} + (0 + 3 \times 2y^{2-1})\boldsymbol{j} \\
&= \boldsymbol{i} + 6y\boldsymbol{j}\,.
\end{aligned}$$

Click on the **green** square to return

**Exercise 1(b)** The gradient of the function

$$f(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$$

is given by:

$$\begin{aligned}
\boldsymbol{\nabla} f(x,y) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} = \frac{\partial}{\partial x}(x^2+y^2)^{\frac{1}{2}}\boldsymbol{i} + \frac{\partial}{\partial y}(x^2+y^2)^{\frac{1}{2}}\boldsymbol{j} \\
&= \frac{1}{2}(x^2+y^2)^{\frac{1}{2}-1} \times \frac{\partial}{\partial x}(x^2)\boldsymbol{i} \\
&\quad + \frac{1}{2}(x^2+y^2)^{\frac{1}{2}-1} \times \frac{\partial}{\partial y}(y^2)\boldsymbol{j} \\
&= \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \times 2x^{2-1}\boldsymbol{i} + \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \times 2y^{2-1}\boldsymbol{j} \\
&= (x^2+y^2)^{-\frac{1}{2}}x\boldsymbol{i} + (x^2+y^2)^{-\frac{1}{2}}y\boldsymbol{j} \\
&= \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2}}\boldsymbol{j}\,.
\end{aligned}$$

Click on the **green** square to return

**Exercise 1(c)** The gradient of the function

$$f(x,y,z) = 3x^2\sqrt{y} + \cos(3z) = 3x^2 y^{\frac{1}{2}} + \cos(3z)\,,$$

is given by:

$$\begin{aligned}
\boldsymbol{\nabla} f(x,y,z) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k} \\
&= 3y^{\frac{1}{2}}\frac{\partial}{\partial x}(x^2)\,\boldsymbol{i} + 3x^2\frac{\partial}{\partial y}(y^{\frac{1}{2}})\,\boldsymbol{j} + \frac{\partial}{\partial z}(\cos(3z))\,\boldsymbol{k} \\
&= 3y^{\frac{1}{2}} \times 2x^{2-1}\,\boldsymbol{i} + 3x^2 \times \frac{1}{2}y^{\frac{1}{2}-1}\,\boldsymbol{j} - 3\sin(3z)\,\boldsymbol{k} \\
&= 6y^{\frac{1}{2}}x\,\boldsymbol{i} + \frac{3}{2}x^2y^{-\frac{1}{2}}\,\boldsymbol{j} - 3\sin(3z)\,\boldsymbol{k} \\
&= 6x\sqrt{y}\,\boldsymbol{i} + \frac{3}{2}\frac{x^2}{\sqrt{y}}\,\boldsymbol{j} - 3\sin(3z)\,\boldsymbol{k}\,.
\end{aligned}$$

Click on the **green** square to return

**Exercise 1(d)** The partial derivative of the function

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}\,,$$

with respect to the variable $x$ is

$$\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}-1} \times \frac{\partial(x^2)}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

and similarly the derivatives $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$ are

$$\frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,, \quad \frac{\partial f}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,.$$

Therefore the gradient is

$$\boldsymbol{\nabla} f(x,y,z) = -\frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,.$$

Click on the **green** square to return

**Exercise 1(e)** The gradient of the function

$$f(x,y) = \frac{4y}{(x^2+1)} = 4y(x^2+1)^{-1}\,,$$

is:

$$\begin{aligned}
\boldsymbol{\nabla} f(x,y) &= 4y \times \frac{\partial}{\partial x}(x^2+1)^{-1}\,\boldsymbol{i} + (x^2+1)^{-1} \times \frac{\partial}{\partial y} 4y\,\boldsymbol{j} \\
&= 4y \times (-1)(x^2+1)^{-1-1}\frac{\partial}{\partial x}(x^2+1)\,\boldsymbol{i} + 4(x^2+1)^{-1}\boldsymbol{j} \\
&= -4y(x^2+1)^{-2} \times 2x\,\boldsymbol{i} + \frac{4}{(x^2+1)}\,\boldsymbol{j} \\
&= -\frac{8xy}{(x^2+1)^2}\,\boldsymbol{i} + \frac{4}{(x^2+1)}\,\boldsymbol{j}\,.
\end{aligned}$$

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**Exercise 1(f)** The partial derivatives of the function

$$f(x, y, z) = \sin(x)\mathrm{e}^y \ln(z)$$

are

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(\sin(x))\ \mathrm{e}^y \ln(z) = \cos(x)\,\mathrm{e}^y\ \ln(z)\,, \\
\frac{\partial f}{\partial y} &= \sin(x)\,\frac{\partial}{\partial y}(\mathrm{e}^y)\ \ln(z) = \sin(x)\mathrm{e}^y\ \ln(z)\,, \\
\frac{\partial f}{\partial z} &= \sin(x)\mathrm{e}^y\ \frac{\partial}{\partial z}(\ln(z)) = \sin(x)\,\mathrm{e}^y\ \frac{1}{z}\,.
\end{aligned}$$

Therefore the gradient is

$$\boldsymbol{\nabla} f(x, y, z) = \cos(x)\,\mathrm{e}^y \ln(z)\boldsymbol{i} + \sin(x)\,\mathrm{e}^y \ln(z)\boldsymbol{j} + \sin(x)\,\mathrm{e}^y \frac{1}{z}\,\boldsymbol{k}\,.$$

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**Exercise 2(a)** The directional derivative of the function

$$f = 3x^2 - 3y^2$$

in the unit vector $\boldsymbol{j}$ direction is given by the scalar product $\boldsymbol{j} \cdot \boldsymbol{\nabla} f$. The gradient of the function $f = 3x^2 - 3y^2$ is

$$\boldsymbol{\nabla} f = 6x\boldsymbol{i} - 6y\boldsymbol{j}$$

Therefore the directional derivative in the $\boldsymbol{j}$ direction is

$$\boldsymbol{j} \cdot \boldsymbol{\nabla} f = \boldsymbol{j} \cdot (6x\boldsymbol{i} - 6y\boldsymbol{j}) = -6y$$

and at the point $(1, 2, 3)$ it has the value $-6 \times 2 = -12$.

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**Exercise 2(b)** The directional derivative of the function $f = \sqrt{x^2+y^2}$ in the direction defined by vector $2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$ is given by the scalar product $\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}f$, where the unit vector $\hat{\boldsymbol{n}}$ is

$$\hat{\boldsymbol{n}} = \frac{2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}}{\sqrt{2^2+2^2+1^2}} = \frac{2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}}{\sqrt{9}} = \frac{2}{3}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}+\frac{1}{3}\boldsymbol{k}\,.$$

The gradient of the function $f$ is

$$\boldsymbol{\nabla}f = \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{i}+\frac{y}{\sqrt{x^2+y^2}}\boldsymbol{j}+0\boldsymbol{k} = \frac{x\boldsymbol{i}+y\boldsymbol{j}}{\sqrt{x^2+y^2}}$$

Therefore the required directional derivative is

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}f = \left(\frac{2}{3}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}+\frac{1}{3}\boldsymbol{k}\right)\cdot\left(\frac{x\boldsymbol{i}+y\boldsymbol{j}}{\sqrt{x^2+y^2}}\right) = \frac{2}{3}\,\frac{x+y}{\sqrt{x^2+y^2}}\,.$$

At the point $(0,-2,1)$ it is equal to $\dfrac{2}{3}\,\dfrac{0-2}{\sqrt{0^2+(-2)^2}} = \dfrac{2}{3}\times\dfrac{-2}{2} = -\dfrac{2}{3}\,.$

Click on the **green** square to return

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**Exercise 2(c)** The directional derivative of the function

$$f = \sin(x) + \cos(x) + \sin(z)$$

in the direction defined by the vector $\pi\boldsymbol{i} + \pi\boldsymbol{j}$ is given by the scalar product $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} f$, where the unit vector $\hat{\boldsymbol{n}}$ is

$$\hat{\boldsymbol{n}} = \frac{\pi\boldsymbol{i} + \pi\boldsymbol{j}}{\sqrt{\pi^2 + \pi^2}} = \frac{\boldsymbol{i} + \boldsymbol{j}}{\sqrt{2}}\,.$$

The gradient of the function $f$ is

$$\boldsymbol{\nabla} f = \cos(x)\boldsymbol{i} - \sin(y)\boldsymbol{j} + \cos(z)\boldsymbol{k}\,.$$

Therefore the directional derivative is

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} f = \frac{\boldsymbol{i} + \boldsymbol{j}}{\sqrt{2}} \cdot (\cos(x)\boldsymbol{i} - \sin(y) + \cos(z)\boldsymbol{k}) = \frac{\cos(x) - \sin(y)}{\sqrt{2}}$$

and at the point $(\pi, 0, \pi\,)$ it becomes $\dfrac{\cos(\pi) - \sin(0)}{\sqrt{2}} = -\dfrac{1}{\sqrt{2}}\,.$

Click on the **green** square to return

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# Solutions to Quizzes

**Solution to Quiz:**

The partial derivative of $xyz^x$ with respect to the variable $z$ is

$$\frac{\partial}{\partial z}\left(xyz^x\right) = xy \times \frac{\partial}{\partial z}\left(z^x\right) = xy \times x \times z^{(x-1)} = x^2yz^{(x-1)}$$

End Quiz

**Solution to Quiz:**

Consider the function $f(x, y) = x\cos(y) + y$, its derivative with respect to the variable $x$ is

$$\begin{aligned}
\frac{\partial}{\partial x} f(x, y) &= \frac{\partial}{\partial x}\left(x\cos(y) + y\right) \\
&= \frac{\partial}{\partial x}(x) \times \cos(y) + \frac{\partial}{\partial x}(y) \\
&= 1 \times \cos(y) + 0 = \cos(y)\,.
\end{aligned}$$

End Quiz

**Solution to Quiz:**

The gradient of the function $f(x,y) = x^2y^3$ is given by:

$$\begin{aligned}
\boldsymbol{\nabla} f(x,y) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} \\
&= \frac{\partial}{\partial x}(x^2y^3)\boldsymbol{i} + \frac{\partial}{\partial y}(x^2y^3)\boldsymbol{j} \\
&= \frac{\partial}{\partial x}(x^2) \times y^3\boldsymbol{i} + x^2 \times \frac{\partial}{\partial y}(y^3)\boldsymbol{j} \\
&= 2x^{2-1} \times y^3\boldsymbol{i} + x^2 \times 3y^{3-1}\boldsymbol{j} \\
&= 2xy^3\boldsymbol{i} + 3x^2y^2\boldsymbol{j}\,.
\end{aligned}$$

End Quiz

**Solution to Quiz:** The partial derivative of the scalar function $f(x, y, z) = x^2yz - xy^2z$ with respect to $y$ is

$$\frac{\partial f}{\partial y}(x, y, z) = x^2 - 2xyz\,.$$

Evaluating it at the point $(1, 1, 1)$ gives

$$\frac{\partial f}{\partial y}(1, 1, 1) = 1^2 - 2 \times 1 \times 1 \times 1 = 1 - 2 = -1\,.$$

This is negative and therefore the function $f$ decreases in the $y$ direction at this point.

It may be verified that the function does not decrease in the $y$ direction at any of the other three points. End Quiz

**Solution to Quiz:** The surface is defined by the equation

$$x^2yz = 1.$$

To find its normal at $(1, 1, 1)$ we need to calculate the gradient of the function $f(x, y, z) = x^2yz$:

$$\boldsymbol{\nabla} f = 2xyz\boldsymbol{i} + x^2z\boldsymbol{j} + x^2y\boldsymbol{k}\,.$$

At the point $(1, 1, 1)$ this is

$$\boldsymbol{\nabla} f = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$$

Thus the required normals to the surface are $\pm(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$. Hence (d) is a normal vector to the surface. End Quiz

**Solution to Quiz:** The surface is defined by the equation

$$\cos(x)yz = -1\,.$$

To find its unit normal at the point $(\pi, 1, 1)$, we need to evaluate the gradient of $f(x, y, z) = \cos(x)yz$:

$$\boldsymbol{\nabla} f = -\sin(x)yz\boldsymbol{i} + \cos(x)z\boldsymbol{j} + \cos(x)y\boldsymbol{k}\,.$$

At the point $(\pi, 1, 1)$ this is

$$\boldsymbol{\nabla} f = 0\boldsymbol{i} + (-1)\boldsymbol{j} + (-1)\boldsymbol{k} = -\boldsymbol{j} - \boldsymbol{k}$$

The magnitude of this vector is

$$\sqrt{(-1)^2 + (-1)^2} = \sqrt{2}\,.$$

Therefore the unit normal is

$$\hat{\boldsymbol{n}} = -\frac{1}{\sqrt{2}}\boldsymbol{j} - \frac{1}{\sqrt{2}}\boldsymbol{k}\,.$$

End Quiz

**Solution to Quiz:** The surface is defined by the equation

$$x^2 + y^2 + z^2 = 169\,.$$

To find its unit normal at point $(5, 0, 12)$ we need to evaluate the gradient of $f(x, y, z) = x^2 + y^2 + z^2$:

$$\boldsymbol{\nabla} f = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}\,.$$

At the point $(5, 0, 12)$ this is

$$\boldsymbol{\nabla} f = 2 \times 5\boldsymbol{i} + 0 \times \boldsymbol{j} + 2 \times 12\boldsymbol{k} = 10\boldsymbol{i} + 24\boldsymbol{k}$$

The magnitude of this vector is

$$\sqrt{(2 \times 5)^2 + (2 \times 12)^2} = \sqrt{4 \times (25 + 144)} = 2\sqrt{169} = 2 \times 13\,.$$

Therefore the unit normal is

$$\hat{\boldsymbol{n}} = \frac{5}{13}\boldsymbol{j} + \frac{12}{13}\boldsymbol{k}\,.$$

End Quiz